ON THE THEORY OF INERTIAL SYSTEMS FOR THE SELF-CONTAINED DETERMINATION OF THE COORDINATES OF A MOVING OBJECT

(K TEORII INERTSIAL'NYKH SISTEM AVTONOMNOGO OPREDELENIIA KOORDINAT DVIZHUSHCHEGOSIA OB"EKTA)

PMM Vol.28, № 1, 1964, pp.39-50

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(Received October 17, 1963)

The article considers a system analogous to that described in [1] for the self-contained determination of the coordinates of the center of gravity of a moving object and its orientation both with respect to the horizon plane and in azimuth.

For arbitrary motion of an object near the surface of the earth, we derive and analyze the equations for the ideal operation of such a system (the equations of relative equilibrium) and the error equations, i.e. the equations of small oscillations about a position of relative equilibrium. The basic instrument errors of components of the system are taken into account.

The equations considered here are fairly general. The equations and results obtained in [1] to [7] follow from these as special cases.

1. We introduce a right-handed orthogonal coordinate system $O'\xi_*\eta_*\zeta_*$; its origin O' is situated at the center of the earth, the ζ_*-axis is directed along the earth-rotation angular velocity vector $\mathbf{u}$ and the ξ_* and η_* axes are in the plane of the equator, so directed that the trihedron $\xi_*\eta_*\zeta_*$ retains a constant orientation with respect to the fixed stars. If we neglect the orbital motion of the earth, this system of coordinates may be considered inertial.

We introduce the coordinate system $O'\xi_0\eta_0\zeta_0$ bound to the earth ; the ζ_0-axis of this system coincides with the ζ_*-axis and ξ_0-axis is directed along the intersection of the equator with the plane of the Greenwich meridian. We shall assume that the coordinate systems $\xi_*\eta_*\zeta_*$ and $\xi_0\eta_0\zeta_0$ coincide at the zero time.

The position of an arbitrary point in the $\xi_0\eta_0\zeta_0$ coordinate system will be difined by spherical coordinates: the latitude φ , the longitude λ , and the distance R of the point from the center of the earth O'. Then the unit vector $\boldsymbol{\rho}$ in the direction from O', the center of the earth, to the arbitrary point O will be

$$\rho(\varphi, \lambda) = \xi_0 \cos\varphi \cos\lambda + \eta_0 \cos\varphi \sin\lambda + \zeta_0 \sin\varphi \quad (1.1)$$

Here ξ_0, η_0 and ζ_0 are unit vectors of the corresponding axes.

Furthermore, we introduce the coordinate system $O'\xi\eta\zeta$, bound to two directions $\rho_1(\varphi_1, \lambda_1)$ and $\rho_2(\varphi_2, \lambda_2)$ which remain fixed in the $\xi_0\eta_0\zeta_0$ coordinate system, in such a way that

$$\xi = \rho_1, \qquad \eta = \frac{\rho_2 - \rho_1 \cos S_0}{\sin S_0}, \qquad \zeta = \frac{\rho_1 \times \rho_2}{\sin S_0}, \qquad \cos S_0 = \rho_1 \cdot \rho_2 \quad (1.2)$$

The position of the center of gravity O of an object moving in the $\xi\eta\zeta$ coordinate system will be defined by spherical coordinates: the angle S measured in the $\xi\eta$ plane from the ξ-axis toward the η-axis, the angle z measured from the $\xi\eta$ plane toward the ζ-axis, and the distance R of the point O from O', the center of the earth.

We shall attach the Darboux trihedron $Ox_1y_1z_1$ to the point O. Its z_1-axis is directed along the line $O'O$ away from the center of the earth and y_1-axis lies in the plane containing the point O and the axis $O'\zeta$. The arrangement of the $\xi\eta\zeta$ and $x_1y_1z_1$ coordinate systems relative to each other is determined by the table of direction cosines shown at the right.

	ξ	η	ζ
x_1	$-\sin S$	$\cos S$	0
y_1	$-\sin z \cos S$	$-\sin z \sin S$	$\cos z$
z_1	$\cos z \cos S$	$\cos z \sin S$	$\sin z$

(1.3)

If the $\xi\eta\zeta$ and $\xi_0\eta_0\zeta_0$ coordinate systems coincide, then the angles z and S become the geocentric latitude φ and longitude λ, and the $x_1y_1z_1$ trihedron becomes oriented with respect to the cardinal points.

	x_1	y_1	z_1
x_0	$\cos\varepsilon$	$\sin\varepsilon$	0
y_0	$-\sin\varepsilon$	$\cos\varepsilon$	0
z_0	0	0	1

(1.4)

We shall also attach to the center of gravity of the object the trihedron $Ox_0y_0z_0$ obtained from the $x_1y_1z_1$ trihedron by rotating through an angle ϵ about the z_1-axis, defined by the table of direction cosines shown at the left. If v_{x_0}, v_{y_0}, and v_{z_0} are the projections of the absolute velocity of the motion of the point O on the x_0, y_0 and z_0 axes, ω_{x_0}, ω_{y_0}, and ω_{z_0} are the projections of the absolute angular velocity of the $x_0y_0z_0$ trihedron on its axes, then

$$v_{x_0} = R\omega_{y_0}, \qquad v_{y_0} = -R\omega_{x_0}, \qquad v_{z_0} = R^{\cdot} \quad (1.5)$$

Here and hereafter, a dot will be used to denote differentiation with respect to time.

Using (1.2), (1.3) and (1.4), we can express ω_{x_0}, ω_{y_0}, and ω_{z_0} by $z^{\cdot}, S^{\cdot}, \epsilon^{\cdot}$ and by u the angular velocity of the rotation of the earth

$$\omega_{x_0} = (-z^{\cdot} + u(-n_{31}\sin S + n_{32}\cos S))\cos\varepsilon + \\ + (S^{\cdot}\cos z + u(-n_{31}\sin z\cos S - n_{32}\sin z\sin S + n_{33}\cos z))\sin\varepsilon \tag{1.6}$$

$$\omega_{y_0} = -(-z^{\cdot} + u(-n_{31}\sin S + n_{32}\cos S))\sin\varepsilon + \\ + (S^{\cdot}\cos z + u(-n_{31}\sin z\cos S - n_{32}\sin z\sin S + n_{33}\cos z))\cos\varepsilon$$

$$\omega_{z_0} = S^{\cdot}\sin z + u(n_{31}\cos z\cos S + n_{32}\cos z\sin S + n_{33}\sin z) + \varepsilon^{\cdot}$$

Here n_{31}, n_{32} and n_{33} are the cosines of the angles between the ζ_0-axis and the ξ,η,ζ axes ; they are equal [8] to

$$n_{31} = \sin\varphi_1, \quad n_{32} = \frac{\sin\varphi_2 - \sin\varphi_1\cos S_0}{\sin S_0}, \quad n_{33} = \frac{\cos\varphi_2\cos\varphi_1\sin(\lambda_1 - \lambda_2)}{\sin S_0} \tag{1.7}$$

Solving Equations (1.7) for $z^{\cdot}$, $S^{\cdot}$ and $\varepsilon^{\cdot}$ and integrating, we have

$$z = -\int_0^t [\omega_{x_0}\cos\varepsilon - \omega_{y_0}\sin\varepsilon - u(-n_{31}\sin S + n_{32}\cos S)]\,dt + z^{\circ}$$

$$S = \int_0^t \frac{1}{\cos z}[\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon - u(-n_{31}\sin z\cos S - \\ - n_{32}\sin z\sin S + n_{33}\cos z)]\,dt + S^{\circ} \tag{1.8}$$

$$\varepsilon = \int_0^t [\omega_{z_0} - \operatorname{tg} z(\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon) - \\ - \frac{u}{\cos z}(n_{31}\cos S + n_{32}\sin S)]\,dt + \varepsilon^{\circ}$$

If ω_{x_0}, ω_{y_0}, and ω_{z_0} are known as functions of time, we can set up a computing scheme simulating Equations (1.8) and thus find z, S and ε .

2. Let the inertial attitude control (Fig. 1) consist of a platform (the trihedron $Oxyz$) bound to the object by a three-gimbal suspension (not shown in Fig. 1). Three gyroscopes G_1, G_2 and G_3 with kinetic moments H_1, H_2 and H_3 are supported in three-gimbal suspensions on the platform. By applying the moments M_x, M_y and M_z to the gyroscopes, we can make the platform rotate with the angular velocities [1]

$$\omega_x = (M_x / H_1) = m_x, \quad \omega_y = (M_y / H_2) = m_y, \quad \omega_z (M_z / H_3) = m_z \tag{2.1}$$

Three accelerometers [1] a_x, a_y and a_z are rigidly connected to the platform ; their axes of sensitivity are directed along the xyz axes. If we make the xyz and $x_0y_0z_0$ trihedra coincide at time zero, then from the accelerometer readings we can set up the moments M_x, M_y and M_z so that for an arbitrary motion of the object, Equations

$$\omega_x \equiv \omega_{x_0}, \quad \omega_y \equiv \omega_{y_0}, \quad \omega_z \equiv \omega_{z_0} \tag{2.2}$$

will always be satisfied and the trihedra will coincide throughout the time that the object moves.

We shall assume that the sensitive masses of the accelerometers are point masses concentrated at the point O Then, having established the necessary relationship between the coefficient of elasticity of the suspension of the sensitive mass and its magnitude, we can regard the accelerometer readings as being equal to

$$a_x = Q_x + F_x, \quad a_y = Q_y + F_y, \quad a_z = Q_z + F_z \tag{2.3}$$

where Q_x, Q_y and Q_z are the projections of the inertial force (referred to a unit mass) of the translatory motion of the xyz coordinates origin, on the axes of sensitivity of the accelerometers and F_x, F_y and F_z are the projections of the earth's gravitational force acting on a unit mass situated at O. The system of coordinates $O'\xi_*\eta_*\zeta_*$ is considered inertial, consequently

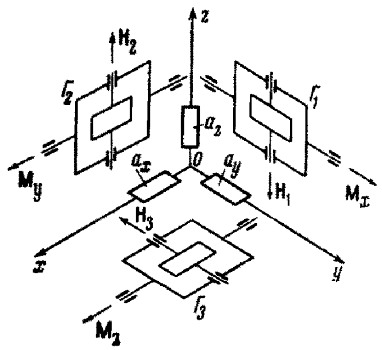

Fig. 1

$$Q_{x_0} = -(R\omega_{y_0})^{\cdot} - R^{\cdot}\omega_{y_0} - R\omega_{x_0}\omega_{z_0}$$
$$Q_{y_0} = (R\omega_{x_0})^{\cdot} - R\omega_{z_0}\omega_{y_0} + \omega_{x_0}R^{\cdot} \tag{2.4}$$
$$Q_{z_0} = -R^{\cdot\cdot} + R\,(\omega_{x_0}^2 + \omega_{y_0}^2)$$

If we neglect the non-central gravitational field of the earth, we have

$$F_{x_0} = F_{y_0} = 0, \qquad F_{z_0} = -k\,/\,R^2 = -g\,(R) \tag{2.5}$$

From (2.3), (2.4) and (2.5) we find

$$\omega_{y_0} = \frac{1}{R}\left[\int_0^t (-a_{x_0} - \omega_{y_0}R^{\cdot} - R\omega_{x_0}\omega_{z_0})\,dt + R^\circ\omega_{y_0}{}^\circ\right] \tag{2.6}$$

$$\omega_{x_0} = \frac{1}{R}\left[\int_0^t (a_{y_0} + R\omega_{z_0}\omega_{y_0} - \omega_{x_0}R^{\cdot})\,dt + R^\circ\omega_{x_0}^\circ\right]$$

$$R^{\cdot} = \int_0^t (-a_{z_0} + R\,(\omega_{x_0}^2 + \omega_{y_0}^2) - g\,(R))\,dt + (R^{\cdot})^\circ, \qquad R = \int_0^t R^{\cdot}\,dt + R^\circ$$

We shall now set up m_{x_0}, and m_{y_0}, simulating equations (2.9), i.e. we shall define m_{x_0}, and m_{y_0} by Equations

$$m_{x_0} = \frac{1}{R}\left[\int_0^t (a_{y_0} + Rm_{z_0}m_{y_0} - m_{x_0}R^{\cdot})\,dt + R^\circ\omega_{x_0}{}^\circ\right]$$
$$m_{y_0} = \frac{1}{R}\left[\int_0^t (-a_{x_0} - m_{y_0}R^{\cdot} - Rm_{x_0}m_{z_0})\,dt + R^\circ\omega_{y_0}{}^\circ\right] \tag{2.7}$$

The quantities R and $R^{\cdot}$ required for setting up the right-hand sides of (2.8), are obtained from the last two equations of (2.6) if we replace ω_{x_0}, and ω_{y_0} in those equations by m_{x_0}, and m_{y_0}; we then obtain

$$R^{\cdot} = \int_0^t (-a_{z_0} + R\,(m_{x_0}{}^2 + m_{y_0}{}^2) - g\,(R))\,dt + (R^{\cdot})^\circ, \quad R = \int_0^t R^{\cdot}\,dt + R^\circ \tag{2.8}$$

It is readily seen that m_{z_0} may be set up as an arbitrary function of time

$$m_{z_0} = m_{z_0}(t) \tag{2.9}$$

Evidently, the moments set up in this manner will identically satisfy

Equations (2.2).

We now make use of (1.8) to find z, S and ε

$$z = -\int_0^t [m_{x_0}\cos\varepsilon - m_{y_0}\sin\varepsilon - u(-n_{31}\sin S + n_{32}\cos S)]\,dt + z^\circ$$

$$S = \int_0^t \frac{1}{\cos z}\,[m \quad \sin\varepsilon + m_{y_0}\cos\varepsilon - u(-n_{31}\sin z\cos S - \\ - n_{32}\sin z\sin S + n_{33}\cos z)]\,dt + S^\circ \tag{2.10}$$

$$\varepsilon = \int_0^t \left[m_{z_0} - \tan z\,(m_{x_0}\sin\varepsilon + m_{y_0}\cos\varepsilon) - \frac{u}{\cos z}(n_{31}\cos S + n_{32}\sin S)\right]dt + \varepsilon^\circ$$

Equations (2.1), (2.3), (2.4), (2.5), (2.7), (2.8), (2.9) and (2.10) form a closed system of equations for ideal (unperturbed) operation of an inertial attitude control.

In (2.5) the gravitational field of the earth was assumed to be central. If we assume that the earth's gravitational force acting on the sensitive mass of an accelerometer lies in a plane containing the axis of rotation of the earth, then for the projections F_{x_2}, F_{y_2}, and F_{z_2} on the axes of a trihedron oriented with respect to the cardinal points (with the Oy_2-axis pointing North), we have

$$F_{x_2} = 0, \qquad F_{y_2} = F_{y_2}(R, \varphi), \qquad F_{z_2} = F_{z_2}(R, \varphi) \tag{2.11}$$

Noting that

$$\cos(y_2\zeta_0)\cos(x_1y_2) = \cos(x_1\zeta_0), \qquad \cos(y_2\zeta_0)\cos(y_1y_2) = \cos(y_1\zeta_0) \tag{2.12}$$

we can use (1.2), (1.3) and (1.4) to find

$$F_{x_0} = (F_{y_2}/\cos\varphi)\,[(-n_{31}\sin S + n_{32}\cos S)\cos\varepsilon + \\ + (-n_{31}\sin z\cos S - n_{32}\sin z\sin S + n_{33}\cos z)\sin\varepsilon]$$

$$F_{y_0} = (F_{y_2}/\cos\varphi)\,[-(-n_{31}\sin S + n_{32}\cos S)\sin\varepsilon + \\ + (-n_{31}\sin z\cos S - n_{32}\sin z\sin S + n_{32}\cos z)\cos\varepsilon] \tag{2.13}$$

$$F_{z_0} = F_{z_2}$$

Consequently, in order to take into account the non-central nature of the earth's gravitational field, we must use Equation (2.13) instead of (2.5). Moreover, since F_{y_2} and F_{z_2} are functions of φ, we must add a relationship between φ and z, S

$$\sin\varphi = n_{31}\cos z\cos S + n_{32}\cos z\sin S + n_{33}\sin z \tag{2.14}$$

As was noted earlier, $m_{z_0}(t)$ may be chosen arbitrarily.

If we assume

$$m_{z_0} = m_{y_0}\tan z - \frac{u}{\cos z}(n_{31}\cos S + n_{32}\sin S), \qquad \varepsilon^\circ = 0 \tag{2.15}$$

then the identity $\varepsilon \equiv 0$ will hold, and the $x_0y_0z_0$ trihedron will coincide with the $x_1y_1z_1$ trihedron the orientation of which is determined by the coordinates of the z, S grid ; if the $\xi\eta\zeta$ and $\xi_0\eta_0\zeta_0$ coordinate systems coincide, the orientation is determined by the coordinates of the φ, λ grid.

Another condition that must be imposed on m_{z_0} to simplify the equations of ideal operation is

$$m_{z_0} = 0 \tag{2.16}$$

The equations for ideal operation in the coordinates z, S or φ, λ for the cases (2.15) and (2.16) may be obtained from the fundamental Equations (2.1), (2.3), (2.7) to (2.10) and (2.13).

We note that if R is a known function of the object coordinates z, S or φ, λ and time, then the a_{z_0} eccelerometer may be omitted. This can happen, for example, in the case of motion on the surface of the ocean, when we may assume that

$$R = a\left(1 - {}^1/_2\, e^2 \sin^2\varphi\right) \tag{2.17}$$

where a is the semi-major axis and e is the eccentricity of Clairaut's ellipsoid. It may also happen in the case of flight near the surface of the earth, when, in addition to (2.17), the altitude above the surface of the earth is measured by means of a radio altimeter. In this case Equations (2.8) drop out of the set of equations for ideal operation, and we add a relation defining R as a function of the coordinates and the altimeter readings.

The above equations for the ideal operation of an inertial attitude control completely determine its nature (within the framework of precession theory) only if all the components of the system are free of error, the initial position and initial angular velocity of the xyz trihedron coincide exactly with the position and velocity of the $x_0 y_0 z_0$ trihedron, and the initial values of the coordinates and their rates of variation, introduced into the computer of the system, are in exact agreement with the coordinates and velocity of the object at the instant the attitude control begins to operate.

If some of these conditions are not satisfied, the motion of the attitude control will naturally de different from the described by the equations for ideal operation, the xyz trihedron will not coincide with the $x_0 y_0 z_0$ trihedron, and the coordinates obtained for the object will contain errors.

3. We shall now derive the error equations. In the derivation we shall take into account only the instrument errors of the system : the accelerometer errors Δa_{x_0}, Δa_{y_0}, Δa_{z_0}, the moments Δm_{x_0}, Δm_{y_0}, Δm_{z_0}, producing free drift in the gyroscopes, and the error δm_{z_0} of the formulation of m_{z_0}. It can be shown that the instrument errors of any part of the system may be reduced to a set of equivalent fundamental errors.

Let the perturbed position of the trihedron $Oxyz$ with respect to the $Ox_0 y_0 z_0$ trihedron be defined by the small angles α, β and γ , so that the direction cosines defining the position of these trihedra with rspect to each other form the table shown here at the right.

	x	y	z
x_0	1	$-\gamma$	β
y_0	γ	1	$-\alpha$
z_0	$-\beta$	α	1

(3.1)

Then the differences between the projections of the absolute angular velocity of the $Oxyz$ trihedron in perturbed and unperturbed motion are

$$\delta\omega_{x_0} = \alpha^{\cdot} + \omega_{y_0}\gamma - \omega_{z_0}\beta, \quad \delta\omega_{y_0} = \beta^{\cdot} - \omega_{x_0}\gamma + \omega_{z_0}\alpha, \quad \delta\omega_{z_0} = \gamma^{\cdot} + \omega_{x_0}\beta - \omega_{y_0}\alpha \tag{3.2}$$

The variations of the accelerometer readings are equal, respectively

$$\delta a_{x_0} = a_{y_0}\gamma - a_{z_0}\beta + \Delta a_{x_0}, \quad \delta a_{y_0} = -a_{x_0}\gamma + a_{z_0}\alpha + \Delta a_{y_0}$$
$$\delta a_{z_0} = a_{x_0}\beta - a_{y_0}\alpha + \Delta a_{z_0} \tag{3.3}$$

From (2.1) we have

$$\delta\omega_{x_0} = \delta m_{x_0} + \Delta m_{x_0}, \qquad \delta\omega_{y_0} = \delta m_{y_0} + \Delta m_{y_0}, \qquad \delta\omega_{z_0} = \delta m_{z_0} + \Delta m_{z_0} \tag{3.4}$$

Finally, varying Equations (2.7) and (2.8), we find

$$\delta m_{x_0} = \frac{1}{R}\Big[\int_0^t (\delta a_{y_0} + \delta R m_{z_0} m_{y_0} + R m_{z_0}\delta m_{y_0} + R m_{y_0}\delta m_{z_0} -$$
$$- R^{\cdot}\delta m_{x_0} - \delta R^{\cdot} m_{x_0})dt + R^\circ\delta\omega^\circ_{x_0} + \delta R^\circ \omega_{x_0}{}^\circ - \delta R m_{x_0}\Big]$$

$$\delta m_{y_0} = \frac{1}{R}\Big[\int_0^t (-\delta a_{x_0} - R^{\cdot}\delta m_{y_0} - \delta R^{\cdot} m_{y_0} - \delta R m_{x_0} m_{z_0} - R m_{x_0}\delta m_{z_0} - \tag{3.5}$$
$$- R m_{z_0}\delta m_{x_0})\, dt + R^\circ\delta\omega_{y_0}{}^\circ + \delta R^\circ\omega_{y_0}{}^\circ - \delta R m_{y_0}\Big] \qquad (\delta m_{z_0} = \delta m_{z_0}(t))$$

$$\delta R^{\cdot} = \int_0^t [-\delta a_{z_0} + \delta R\,(m_{x_0}{}^2 + m_{y_0}{}^2) + \qquad \left(\delta R = \int_0^t \delta R^{\cdot}\, dt + \delta R^\circ\right)$$
$$+ 2R\,(m_{x_0}\delta m_{x_0} + m_{y_0}\delta m_{y_0}) - \delta g\,(R)]\, dt + \delta\,(R^{\cdot})^\circ$$

In the variation of Equations (2.7), the variations of the corrections for the non-central nature of earth's gravitational field were neglected as being small. The variations are isochronous, the time is not varied, and the timer aboard the platform is thus assumed to be ideal. The quantities $\delta\omega_{y_0}{}^\circ$, $\delta\omega_{x_0}{}^\circ$, δR° and $\delta(R^{\cdot})^\circ$ denote the input errors in the initial data.

From (3.2) to (3.5), noting that

$$\delta g\,(R) = -2g\delta R\,/\,R \tag{3.6}$$

bearing in mind the equations of ideal operation, performing the change of variables

$$x = R\alpha, \qquad y = R\beta \tag{3.7}$$

and introducing the notation

$$\omega_0{}^2 = g\,/\,R \tag{3.8}$$

we obtain the following equations for determining x, y, δR and γ :

$$\begin{aligned} x^{\cdot\cdot} + (\omega_0{}^2 - \omega_{x_0}{}^2 - \omega_{z_0}{}^2)\, x - (\omega_{x_0}\omega_{y_0} + \omega_{z_0}{}^{\cdot})\, y - \\ - 2\omega_{z_0} y^{\cdot} + (\omega_{x_0}{}^{\cdot} - \omega_{y_0}\omega_{z_0})\,\delta R + 2\omega_{x_0}\delta R^{\cdot} = \\ = \Delta a_{y_0} + R\,\Delta m_{x_0}{}^{\cdot} + 2R^{\cdot}\,\Delta m_{x_0} - R\omega_{z_0}\Delta m_{y_0} - R\omega_{y_0}\Delta m_{z_0} \\ y^{\cdot\cdot} + (\omega_0{}^2 - \omega_{y_0}{}^2 - \omega_{z_0}{}^2)\, y - (\omega_{x_0}\omega_{y_0} - \omega_{z_0}{}^{\cdot})\, x + 2\omega_{z_0} x^{\cdot} + \\ + (\omega_{x_0}\omega_{z_0} + \omega_{y_0}{}^{\cdot})\,\delta R + 2\omega_{y_0}\delta R^{\cdot} = \\ = -\Delta a_{x_0} + R\,\Delta m_{y_0}{}^{\cdot} + 2R^{\cdot}\,\Delta m_{y_0} + R\omega_{z_0}\Delta m_{x_0} + R\omega_{x_0}\Delta m_{z_0} \end{aligned} \tag{3.9}$$

$$\delta R^{\cdot\cdot} - (2\omega_0^2 + \omega_{x_0}^2 + \omega_{y_0}^2)\,\delta R - (\omega_{x_0}^{\cdot} + \omega_{y_0}\omega_{z_0})\,x - (\omega_{y_0}^{\cdot} - \omega_{x_0}\omega_{z_0})\,y - $$
$$- 2\omega_{x_0}x^{\cdot} - 2\omega_{y_0}y^{\cdot} = -\Delta a_{z_0} - 2R\omega_{x_0}\Delta m_{x_0} - 2R\omega_{y_0}\Delta m_{y_0} \qquad (3.9)$$

$$\gamma^{\cdot} = -\omega_{x_0}\beta + \omega_{y_0}\alpha + \delta m_{z_0} + \Delta m_{z_0} \qquad (3.10)$$

The initial conditions for Equations (3.9) and (3.10) will be

$$x^\circ = R^\circ\alpha^\circ, \quad y^\circ = R^\circ\beta^\circ, \quad \delta R^\circ, \quad \delta(R^{\cdot})^\circ, \quad \gamma^\circ$$
$$(x^{\cdot})^\circ = (R^{\cdot})^\circ\alpha^\circ + R^\circ(\delta\omega_{x_0}{}^\circ + \Delta m_{x_0}{}^\circ - \omega_{y_0}{}^\circ\gamma^\circ + \omega_{z_0}{}^\circ\beta^\circ) \qquad (3.11)$$
$$(y^{\cdot})^\circ = (R^{\cdot})^\circ\beta^\circ + R^\circ(\delta\omega_{y_0}{}^\circ + \Delta m_{y_0}{}^\circ + \omega_{x_0}{}^\circ\gamma^\circ - \omega_{z_0}{}^\circ\alpha^\circ)$$

We proceed to formulate the equations for the errors in the coordinates z, S and the azimuth angle ε. Varying Equations (2.10), we obtain

$$\delta z^{\cdot} = -\delta m_{x_0}\cos\varepsilon + \delta m_{y_0}\sin\varepsilon + (m_{x_0}\sin\varepsilon + m_{y_0}\cos\varepsilon)\,\delta\varepsilon - \qquad (3.12)$$
$$- u\,(n_{31}\cos S + n_{32}\sin S)\,\delta S$$

$$\delta S^{\cdot}\cos z = \delta m_{x_0}\sin\varepsilon + \delta m_{y_0}\cos\varepsilon +$$
$$+ (m_{x_0}\cos\varepsilon - m_{y_0}\sin\varepsilon)\,\delta\varepsilon + \tan z\,(m_{x_0}\sin\varepsilon + m_{y_0}\cos\varepsilon)\,\delta z +$$
$$+ u\sin z\,(-n_{31}\sin S + n_{32}\cos S)\,\delta S + u\sec z\,(n_{31}\cos S + n_{32}\sin S)\,\delta z$$

$$\delta\varepsilon^{\cdot}\cos z = \delta m_{z_0}\cos z - \sin z\,(\delta m_{x_0}\sin\varepsilon + \delta m_{y_0}\cos\varepsilon) -$$
$$- \sin z\,(m_{x_0}\cos\varepsilon - m_{y_0}\sin\varepsilon)\,\delta\varepsilon + u\,(n_{31}\sin S - n_{32}\cos S)\,\delta S -$$
$$- [\sec z\,(m_{x_0}\sin\varepsilon + m_{y_0}\cos\varepsilon) + u\tan z\,(n_{31}\cos S + n_{32}\sin S)]\,\delta z$$

By introducing the new variables α_1, β_1 and γ_1

$$\alpha_1 = -\delta z\cos\varepsilon + \delta S\cos z\sin\varepsilon, \quad \beta_1 = \delta z\sin\varepsilon + \delta S\cos z\cos\varepsilon$$
$$\gamma_1 = \delta\varepsilon + \delta S\sin z \qquad (3.13)$$

we can transform Equations (3.12) to the form

$$\alpha_1 = \delta m_{x_0} + \beta_1 m_{z_0} - \gamma_1 m_{y_0}, \qquad \beta_1^{\cdot} = \delta m_{y_0} - \alpha_1 m_{z_0} + \gamma_1 m_{x_0}$$
$$\gamma_1^{\cdot} = \delta m_{z_0} - \beta_1 m_{x_0} + \alpha_1 m_{y_0} \qquad (3.14)$$

It is readily seen that the variables α_1, β_1 and γ_1 represent the angular errors in the determination of the coordinates and the azimuth.

From (3.4) and (3.14), again introducing new variables

$$\alpha_2 = \alpha_1 - \alpha, \qquad \beta_2 = \beta_1 - \beta, \qquad \gamma_2 = \gamma_1 - \gamma \qquad (3.15)$$

we obtain a second group of equations for the error of the inertial attitude control

$$\alpha_2^{\cdot} - \omega_{z_0}\beta_2 + \omega_{y_0}\gamma_2 = -\Delta m_{x_0}, \qquad \beta_2^{\cdot} - \omega_{x_0}\gamma_2 + \omega_{z_0}\alpha_2 = -\Delta m_{y_0} \qquad (3.16)$$
$$\gamma_2^{\cdot} - \omega_{y_0}\alpha_2 + \omega_{x_0}\beta_2 = -\Delta m_{z_0}$$

The initial conditions for the differential equations (3.16) are found from (3.13) and (3.15).

If there is no accelerometer along the z_0-axis and information on the magnitude of R is given in addition, then the third equation drops out of the first group (3.9) of error equations. If R is given either as a constant or as a function of time, then δR and $\delta R^{\cdot}$ which appear in the first two equations of (3.9), will be known (possibly random) functions of time. In the case where R and $R^{\cdot}$ are determined by the attitude control as functions of the other two coordinates of the object, we must consider also the equations obtained by varying these functions. Thus, if the motion takes place on the surface of the ocean, we have

$$\delta R = \delta R(\varphi) + \Delta R \tag{3.17}$$

where $\delta R(\varphi)$ is obtained by varying (2.17) and (2.14), and ΔR is the instrument error.

The error equations (3.9), (3.10), (3.15) and (3.16) are fairly general. They constitute the error equations of an arbitrary inertial system for the selfcontained determination of the coordinates of an object by means of accelerometers and gyroscopes. They are essentially obtained directly from Newton's laws. The concrete system considered here was used only as a framework for deriving them, and no parameters of this system appear on the left-hand sides of (3.9), (3.10), (3.15) and (3.16).

It will be shown below that the special cases of these equations will include the equations for the small oscillations of a physical pendulum with an extended length equal to the radius of the earth [2] and [9], the equations of a two-gyroscope pendulum [4] and [6], and the equations of a gyro-horizon compass (north-seeking gyroscope) studied in [3,5,6 and 7].

4. We shall prove that Equations (3.9), (3.10), (3.15) and (3.16) make possible a group of rotations by an arbitrary angle $\vartheta(t)$ about the Oz_0-axis. This property follows from the fact that $m_{z_0}(t)$ is arbitrary, and consequently so is the angle ε characterizing the orientation of the trihedron in azimuth. This also may be proved directly. In equations (3.9), (3.10) and (3.16) we adopt the new variables x', y', $\delta R'$, γ', $(\delta R')^{\cdot}$, α_1', α_2', β_1', β_2', γ_1', γ_2' by the nonsingular linear transformation

$$\begin{gathered} x = x'\cos\vartheta - y'\sin\vartheta, \qquad y = x'\sin\vartheta + y'\cos\vartheta \\ \alpha_1 = \alpha_1'\cos\vartheta - \beta_1'\sin\vartheta, \qquad \alpha_2 = \alpha_2'\cos\vartheta - \beta_2'\sin\vartheta \\ \beta_1 = \alpha_1'\sin\vartheta + \beta_1'\cos\vartheta, \qquad \beta_2 = \alpha_2'\sin\vartheta + \beta_2'\cos\vartheta \\ \gamma = \gamma', \qquad \gamma_1 = \gamma_1', \qquad \gamma_2 = \gamma_2', \qquad \delta R = \delta R', \qquad \delta R^{\cdot} = (\delta R')^{\cdot} \end{gathered} \tag{4.1}$$

where $\vartheta(t)$ is an arbitrary function of time. The transformation inverse to (4.1) is obvious. The transformation (4.1) converts Equations (3.9), (3.10) and (3.16) into equations in new variables with the newly derived equations retaining the same form as the original equations. The coefficients and the right-hand sides are transformed as follows:

$$\begin{gathered} \omega_{x_0}' = \omega_{x_0}\cos\vartheta + \omega_{y_0}\sin\vartheta, \quad \omega_{y_0}' = -\omega_{x_0}\sin\vartheta + \omega_{y_0}\cos\vartheta, \quad \omega_{z_0}' = \omega_{z_0} + \vartheta^{\cdot} \\ R' = R, \qquad (R')^{\cdot} = R^{\cdot}, \qquad (\omega_0^2)' = \omega_0^2 \\ \Delta a_{x_0}' = \Delta a_{x_0}\cos\vartheta + \Delta a_{y_0}\sin\vartheta, \quad \Delta a_{y_0}' = -\Delta a_{x_0}\sin\vartheta + \Delta a_{y_0}\cos\vartheta \\ \Delta a_{z_0}' = \Delta a_{z_0}, \qquad \Delta m_{z_0}' = \Delta m_{z_0}, \qquad \delta m_{z_0}' = \delta m_{z_0} \\ \Delta m_{x_0}' = \Delta m_{x_0}\cos\vartheta + \Delta m_{y_0}\sin\vartheta, \qquad \Delta m_{y_0}' = -\Delta m_{x_0}\sin\vartheta + \Delta m_{y_0}\cos\vartheta \end{gathered} \tag{4.2}$$

To prove this, we substitute (4.1) and (4.2) into (3.9), (3.10), (3.15) and (3.16). Substitution into the third equation of (3.9), into Equation (3.10) and into the last equations of (3.15) and (3.16) immediately shows the validity of the foregoing statement for these equations. After substituting into the first and second equations of (3.9), we must multiply them by $\cos\vartheta$ and $\sin\vartheta$, respectively, and add the results; we then obtain the first equation of the new system. If we multiply by $-\sin\vartheta$ and $\cos\vartheta$, respectively, and then add, we obtain the second equation. In a similar manner we can obtain the new equations for α_1', β_1' and α_2', β_2' from the first two equations of (3.15) and (3.16).

From (4.1) and (4.2) we see that the inverse transformation is obtained from the direct transformation if we replace ϑ by $-\vartheta$; finally, we can readily establish that the two successive transformations ϑ_1 and ϑ_2 are equivalent to one transformation such that $\vartheta_3 = \vartheta_1 + \vartheta_2$.

The above property of Equations (3.9), (3.10), (3.15) and (3.16) enables us, in the analysis of this system, to select the accompanying trihedron $Ox_0y_0z_0$ in different ways for different laws of object motion. In a number of cases it is convenient to use a trihedron one of whose axes, for example x_0, lies in the plane containing the object's absolute velocity vector and the center of the earth. The angle $\vartheta(t)$ is obviously found from the condition $\omega_{x_0}' = 0$, which yields

$$\tan\vartheta = \omega_{x_0} / \omega_{y_0} \tag{4.3}$$

The equations for this case are obtained from (3.9), (3.10), (3.15) and (3.16) for $\omega_{x_0} = 0$.

In most cases the problem of a navigational system includes the determination of object coordinates with respect to the earth; for this reason, the suitable choice for the angle ϑ will be one for which one of the axes lies in the plane containing the relative velocity vector and the center of the earth.

It is also found useful to employ an azimuthally free trihedron in which ϑ is found from the condition $\omega_{z_0}' = 0$; hence

$$\vartheta = -\int_0^t \omega_{z_0}\, dt \tag{4.4}$$

Appropriate equations are obtained from the system (3.9), (3.10), (3.15) and (3.16) for $\omega_{z_0} = 0$. Since in this case the equations are independent of ω_{z_0}, while ω_{x_0} and ω_{y_0} are limited by the upper velocity limit of the object, it is sometimes easier to analyze equations in this form.

If ϑ is so chosen that the $x_0y_0z_0$ trihedron becomes oriented with respect to the cardinal points, it is convenient to analyze equations (3.9), (3.10), (3.15) and (3.16), for an object which is motionless with respect to the earth or which moves along a parallel.

If there is no accelerometer along the z_0-axis, then for $\Delta a_{x_0} = \Delta a_{y_0} = \Delta m_{x_0} = \Delta m_{y_0} = \Delta m_{z_0} = 0$, the first two equations of (3.9) become the equations for the small oscillations of a special physical pendulum [2] and [9]. The left-hand sides of these equations are of the form [2]

$$x^{\cdot\cdot} + (\omega_0^2 - \omega_{x_0}^2 - \omega_{z_0}^2)\,x - (\omega_{x_0}\omega_{y_0} + \omega_{z_0}^{\cdot})\,y - 2\omega_{z_0}y^{\cdot} = 0$$

$$y^{\cdot\cdot} + (\omega_0^2 - \omega_{y_0}^2 - \omega_{z_0}^2)\,y - (\omega_{x_0}\omega_{y_0} - \omega_{z_0}^{\cdot})\,x + 2\omega_{z_0}x^{\cdot} = 0 \qquad (4.5)$$

The equations for the small oscillations of a two-gyroscope pendulum [4], the Anschütz-Geckeler gyro-horizon compass [3] and the system for the self-contained determination of object coordinates [1] reduce to Equations (4.5).

For example, the equations for small oscillations of a gyro-horizon compass before simplification [3] have the form

$$-ml\,(v\alpha)^{\cdot} + lF\beta = -\omega\,2B\delta\sin\sigma, \qquad \beta^{\cdot} + (v\,/\,R)\,\alpha = \omega\,\gamma \qquad (4.6)$$

$$\gamma + (2B\delta\sin\sigma)\,/\,mlR = -\omega\,\beta, \qquad -(2B\delta\sin\sigma)^{\cdot} + l\,(F - mv^2\,/\,R)\,\gamma = \omega\,ml\alpha$$

Here β and γ are the angles by which the platform of the gyro-horizon compass deviates from the edges of the Darboux trihedron $Ox_0y_0z_0$, whose Ox_0-axis is directed along the vector of the absolute velocity v of motion of the object and m, l, B and $R = R^\circ$ are constants,

$$v = \omega_{y_0}R, \qquad \omega = \omega_{z_0}, \qquad F = mg_0 \qquad (4.7)$$

Eliminating α and $2B\delta\sin\sigma$ from (4.6) and making use of (4.7), we obtain

$$\beta^{\cdot\cdot} + \omega_0^2\beta = \omega_{z_0}^2\beta + \omega_{z_0}^{\cdot}\gamma + 2\omega_{z_0}\gamma^{\cdot}, \quad \gamma^{\cdot\cdot} + \omega_0^2\gamma = (\omega_{y_0}^2 + \omega_{z_0}^2)\,\gamma - \omega_{z_0}^{\cdot}\beta - 2\omega_{z_0}\beta^{\cdot} \quad (4.8)$$

Since, in the case under consideration, in equations (4.5) we should set $\omega_{x_0} = 0$ and the β,γ of Equation (4.8) correspond to the α,β of Equation (4.6), it follows that these equations are identical. The identity of the equations of a two-gyroscope pendulum [4] and the system considered in [1] is proved in a similar manner.

In [1], [3] and [4] a solution is given for the simplified equations (4.6), using a complex-valued formulation. The simplification introduced consists of the fact that in (4.8) v is considered equal to zero. This is equivalent to the simplification $\omega_{x_0} = \omega_{y_0} = 0$ in Equations (4.5). Writing Equations (4.5) with respect to an azimuthally free trihedron, we obtain in this case

$$(\alpha')^{\cdot\cdot} + \omega_0^2\alpha' = 0, \qquad (\beta')^{\cdot\cdot} + \omega_0^2\beta' = 0 \qquad (4.9)$$

The solution of Equations (4.9) is obvious.

It must be noted that [1] and [2] pointed out to the equivalence of Equations (4.6) and (4.5), and [2], in addition, showed that the form of Equations (4.5) is retained under the transformation (4.1) and (4.2), and noted that neglecting ω_{y_0} in (4.8) reduces the equations to the harmonic equations (4.9). Apparently, these comments escaped attention, and therefore in [5] and [6] considerable effort was wasted in proving that in the case $\omega_{y_0} = 0$ Equations (4.8) can be reduced to equations with constant coefficients.

The stability of the system (4.6) for constant ω and v and small values of the variables is investigated in [7]. The Liapunov stability condition, derived in these studies

$$\omega_0^2 - \omega_{y_0}^2 - \omega_{z_0}^2 > 0 \qquad (4.10)$$

may also be obtained by investigating the characteristic equations of the system (4.8) by Hurwitz's method if an arbitrarily small total dissipation is introduced into this system.

The three equations (3.9) form a closed system and may be considered independent of the others. Together with (3.7), they determine the angular oscillations of the attitude control platform with respect to the $Ox_0y_0z_0$ trihedron in the angles α and β, and also the quantity δR. If from these we find x, y and δR, then we can use (3.7) to determine by quadratures the angle γ of the azimuthal oscillations of the platform from Equation (3.10). Equations (3.16) also form a closed system. If we find from these the angles α_2, β_2 and γ_2, and make use of the solutions x and y of Equations (3.9) and also (3.7) and (3.15), we can obtain solutions for α_1, β_1 and γ_1 which determine the errors in the azimuth angle and the coordinates computed by the attitude control system.

In the general case (3.9), (3.10) and (3.16) are equations with variable coefficients. Their right-hand sides may be either explicite or random functions of time. For this reason, the study of these equations involves considerable difficulty.

Only in a few special cases can equations (3.9), (3.10) and (3.16) be reduced to equations with constant coefficients. These cases include : a base which is motionless in the coordinate system $O'\xi_*\eta_*\zeta_*$ (in this case may be any arbitrary function of time); motion at constant velocity at a fixed distance from the center of the earth in a plane passing through that center; and motion at constant velocity along a parallel.

Plane motion at a constant distance from the center of the earth, for a case in which the object velocity varies in a specified manner, reduces the system (3.9) to the Mathieu-Hill equation.

For the general case of object motion the system (3.16) coincides with the equations that determine the direction cosines n_{ij} of the axes of the trihedron $Ox_0y_0z_0$ in the coordinate system $O'\xi_*\eta_*\zeta_*$ on the basis of specified values of ω_{x_0}, ω_{y_0}, ω_{z_0}. This system has a first integral and reduces to the Riccati equation [10]. However, if the object motion is specified in such a way that the n_{ij} are known functions of time, the system (3.16) may be integrated to the end.

The analysis of Equations (3.9), (3.10) and (3.16) may be facilitated by the fact that these are equations in variations, so that no great accuracy is required in their solution. For this reason, various approximate methods of calculation may be used. Different methods may be found suitable for different classes of object motion. Examples of such classes of motion may be found in the motion of sea-going ships, the motion of aircraft in the atmosphere and Keplerian or nearly Keplerian motion.

The first of these classes of motion is characterized by the fact that the velocity v_1 with respect to the earth is low in comparison with the circumferential velocity of motion of points of the Equator,

$$|(v_1/R) \ll u \tag{4.11}$$

and that the change in distance from the center of the earth, caused by the fact that the earth is not a perfect sphere, is small. In this class of motion the amount of time spent in continuous operation may be large. For this case the initial solutions to be refined by approximation methods may be taken to be the solutions for the case of a base motionless with respect to the earth.

For the second class of motion the velocity is considerably greater than the circumferential velocity of rotation of the earth, but much smaller than the circular orbital velocity, i.e.

$$u^2 \ll \omega_{x_0}{}^2 \ll \omega_0{}^2, \qquad u^2 \ll \omega_{y_0}{}^2 \ll \omega_0^2 \tag{4.12}$$

the amount of change in R is small and the time required for continuous operation is small in comparison with the first case. For this class the initial solution may be taken to be that for the case of a base motionless in absolute space or else for the case of plane motion in the coordinate system $O'\xi_*\eta_*\zeta_*$.

Finally, for Keplerian and nearly Keplerian motion when ω_{x_*} and ω_{y_*} are comparable to ω_0, the initial solutions used may be those for the case of plane motion, and for Keplerian motion with a small orbital eccentricity we may use the case of plane motion at a constant distance from the center of the earth (a circular orbit).

In some cases it is useful to specify Equations (3.9), (3.10), (3.15) and (3.16) as projections on the axes of the coordinate system $O'\xi_*\eta_*\zeta_*$

$$(\delta\xi_*')^{\cdot\cdot} + \frac{\omega_0^2}{R^2}\left[(\eta_*{}^2 + \zeta_*{}^2 - 2\xi_*{}^2)\,\delta\xi_*' - 3\xi_*\eta_*\delta\eta_*' - 3\xi_*\zeta_*\delta\zeta_*'\right] = \tag{4.13}$$

$$= -\Delta a_\xi{}^* - 2\eta_*{}^\cdot\Delta m_\zeta{}^* + 2\zeta_*{}^\cdot\Delta m_\eta{}^* - \eta_*(\Delta m_\xi{}^*)^\cdot + \zeta_*(\Delta m_\eta{}^*)^\cdot$$

$$(\delta\eta_*')^{\cdot\cdot} + \frac{\omega_0^2}{R^2}\left[(\zeta_*{}^2 + \xi_*{}^2 - 2\eta_*{}^2)\,\delta\eta_*' - 3\eta_*\zeta_*\delta\zeta_*' - 3\eta_*\xi_*\delta\xi_*'\right] =$$

$$= -\Delta a_\eta{}^* - 2\zeta_*{}^\cdot\Delta m_\zeta{}^* + 2\xi_*{}^\cdot\Delta m_\zeta{}^* - \zeta_*(\Delta m_\zeta{}^*)^\cdot + \xi_*(\Delta m_\zeta{}^*)^\cdot$$

$$(\delta\zeta'_*)^{\cdot\cdot} + \frac{\omega_0^2}{R^2}\left[(\xi_*{}^2 + \eta_*{}^2 - 2\zeta_*{}^2)\,\delta\zeta_*' - 3\xi_*\zeta_*\delta\xi_*' - 3\eta_*\zeta_*\delta\eta_*'\right] =$$

$$= -\Delta a_\zeta{}^* - 2\xi_*{}^\cdot\Delta m_\eta{}^* + 2\eta_*{}^\cdot\Delta m_\xi{}^* - \xi_*(\Delta m_\eta{}^*)^\cdot + \eta_*(\Delta m_\xi{}^*)^\cdot$$

$$R^2 = \xi_*{}^2 + \eta_*{}^2 + \zeta_*{}^2$$

In order to find the errors in the coordinates, Equations

$$\delta\xi_* = \delta\xi_*' + \delta\xi_*'', \quad \delta\eta_* = \delta\eta_*' + \delta\eta_*'', \quad \delta\zeta_* = \delta\zeta_*' + \delta\zeta_*'' \tag{4.14}$$

must be supplemented by the relations (4.15)

$$\delta\xi_*'' = \eta_*\gamma_* - \zeta_*\beta_*, \quad \delta\eta_*'' = -\xi_*\gamma_* + \zeta_*\alpha_*, \quad \delta\zeta_*'' = \xi_*\beta_* - \eta_*\alpha_*$$

where α_*, β_* and γ_* are found from Equations

$$\alpha_*^\cdot = \Delta m_{\xi_*}, \quad \beta_*^\cdot = \Delta m_{\eta_*}, \quad \gamma_*^\cdot = \Delta m_{\zeta_*} \tag{4.16}$$

From (4.16) and (3.16) it follows that (3.16) is integrable when the n_{ij} are given.

The structure of Equations (4.13) to (4.16) is similar to Equations (3.9), (3.10), (3.15) and (3.16). They may be either obtained by projecting Equations (3.9), (3.10), (3.15) and (3.16) on the axes of $O'\xi_*\eta_*\zeta_*$ or derived directly as the equations for the errors in an inertial system in which the accelerations are measured along the direction $O\xi_*$, $O\eta_*$ and $O\zeta_*$, which have a fixed orientation in space. Equations (4.13), (4.14), (4.15) and (4.16) retain their form when a change is made to an arbitrary trihedron $\xi_*^\circ\,\eta_*^\circ\,\zeta_*^\circ$ whose orientation with respect to $\xi_*\eta_*\zeta_*$ remains fixed.

The author is grateful to A.Iu. Ishlinskii and V.N. Koshliakov for their comments on the present work.

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Translated by A.S.